Functional analysis
List 2

Ex 1. Prove that convergence of sequence of elements of $C[a,b]$ in the norm $\|x\|_\infty = \sup_{t\in[a,b]} |x(t)|$ is equivalent to the uniform convergence of a sequence of functions.

Ex 2. Show that the pair $(C^{(1)}[a,b], \|\cdot\|_\infty)$ where $\|x\|_\infty = \sup_{t\in[a,b]} |x(t)|$ is not a Banach space?

Ex 3. Check whether the pair $(C[a,b], \|\cdot\|_1)$ where $\|x\|_1 = \int_a^b |x(t)|\,dt$ is a Banach space?

Ex 4. Check whether the subset M of $C[0,1]$ is open, closed or bounded.

N	M	N	M
1.	$\{x : x(0) = 0\}$	3.	$\{x : \int_0^1 x(t)dt = 0\}$
2.	$\{x : x(0) = x(1)\}$	4.	$\{x : \int_0^1 x(t)dt < 1\}$

Ex 5. Check if a sequence of functions x_n converges to α in $C[a,b]$ where

N	$C[a,b]$	x_n	α
1.	$C[-3,3]$	$\sqrt{t^2 + \frac{1}{n^3}}$	$\lvert t\rvert$
2.	$C[0,8]$	$(\frac{t}{8})^n - (\frac{t}{8})^{2n} + t$	t
3.	$C[0,1]$	$t^{2n} - t^{n+1} + t$	t
4.	$C[-4,4]$	$\frac{1}{n^2}\sqrt{n^4t^2+1}$	t
5.	$C[1,2]$	$n^2(\sqrt{t + \frac{1}{n^3}} - \sqrt{t})$	$\frac{1}{2\sqrt{t}}$
6.	$C[\frac{1}{2}, \frac{3}{2}]$	$\frac{t^n - t}{1+t^n}$	1
7.	$C[0,1]$	$\sqrt[n]{1+t^n}$	t
9.	$C[0, \frac{1}{3}]$	$3^n t^n - 3^{n+1}t^{n+1} - 3t^n$	0
10.	$C[0,2]$	$\sqrt[n]{1+t^n}$	$\begin{cases} 1, & t \in [0,1] \\ t, & t \in [1,2] \end{cases}$

Ex 6. Examine convergence of a sequence of functions x_n in X where

N	X	x_n
1.	$C[-1,0]$	$\frac{1}{n}\sqrt[3]{n^3t+1}$
2.	$C[1,2]$	$\frac{2t^n-1}{1+t^n}$
3.	$C[-1, \frac{1}{2}]$	$\frac{(t+1)^{2n} - t^2 n}{(t+2)^{2n}}$
4.	$C[1,2]$	$n\sin(\frac{t^2}{n}) + \frac{t^3}{n}$
5.	$C[0,9]$	$\frac{9^n t^n - t^{2n}}{9^{2n}}$
6.	$C^{(1)}[0,1]$	$\frac{t^n}{n}$
7.	$C[-1,5]$	$\operatorname{arctg}(n(t^2+1))$
8.	$C[-\frac{\pi}{2}, 0]$	$(\sin t)^{2n} + \sqrt[3]{\frac{t}{n}}$
9.	$C[1,2]$	$\frac{t^2}{n^2}\ln(\frac{t}{n})$